Stabilization of Computational Procedures for Constrained Dynamical Systems

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A new stabilization method of treating constraints in multibody dynamical systems is presented. By tailoring a penalty form of the constraint equations, the method achieves stabilization without artificial damping and yields a companion matrix differential equation for the constraint forces; hence, the constraint forces are obtained by integrating the companion differential equation for the constraint forces in time. A principal feature of the method is that the errors committed in each constraint condition decay with its corresponding characteristic time scale associated with its constraint force. Numerical experiments indicate that the method yields a marked improvement over existing techniques.

I. Introduction

THE dynamics of flexible multibody systems, such as the design of robotic manipulators, mechanical chains, and satellites, is becoming increasingly important in engineering. Computer simulation of such multibody dynamical (MBD) systems requires a concerted integration of several computational aspects. These include selection of a data structure for describing the system topology, computerized generation of the governing equations of motion, incorporation of constraint conditions, implementation of suitable solution algorithms, and easy interpretation of the simulation results.

Traditionally, the task of formulating the equations of motion has been of dominant concern to many dynamists. As a result, several MBD formulations have been proposed; these differ primarily in the manner in which they incorporate constraints and in their resulting system topologies.[1–13] Hence, reliability and cost of existing MBD simulation packages have been strongly affected by how well the equations of motion have been streamlined and how well the constraints are preserved during the numerical solution stage.

As dynamists face more complex problems, particularly in the field of large space structures, a new consensus is emerging: MBD simulation requires a data structure that can accommodate various system topologies. A primary motivation for espousing a maximum flexibility in the data structure is to allow, for each subsystem of a complex MBD system, the adoption of different modeling assumptions, different formulations of the equations of motion, and different solution techniques. Once this need is recognized, compatibility of subsystems as well as of various constraints becomes a focal computational issue. However, enforcing such subsystem and kinematical compatibilities leads to a formulation that involves a set of auxiliary constraints that must be satisfied at each integration step.

Because it is important in the simulation of MBD systems to treat the resulting constraints accurately and reliably, several computational procedures have been proposed. These include the technique for condensing dependent variables via singular-value decomposition by Walton and Steeves,[14] equilibrium correction strategies by Baumgarte,[15,16] penalty formulation by Orlandea et al.[17] and Lötstedt,[18] the coordinate partitioning technique by Wehage and Haug,[19] and the differential/algebraic approach by Gear[20] and Petzold.[21] In addition, recent reports by Huston and Kamman,[23] Fuehrer and Wallrapp,[24] Schwertassek and Roberson,[25] and Nikravesh[26] address various related techniques.

Among the procedures cited, it is generally agreed that Baumgarte's technique is the most reliable one for handling constraints. Thus, we believe that new methods for constraint stabilization should be compared with Baumgarte's technique. However, an examination of Baumgarte's technique has revealed that it has three important algorithmic and software difficulties.

First, according to Baumgarte's formulation that leads to his constraint stabilization, the error committed in all the constraint conditions during time integration steps can decay only with a uniform characteristic time constant. In other words, each of the constraint equations converges at the same rate regardless of its physical nature. This uniform convergence rate masks an important physical phenomenon: the characteristic time constants of each constraint equation are different, since Lagrangian multipliers associated with the constraint equations exhibit different physical response characteristics. Hence, Baumgarte's technique does not exploit the well-known observation that the principal errors in multi-degree-of-freedom systems behave the same way as do those associated with the individual physical components.

Second, Baumgarte's technique requires that the solution matrix, $B^TM^{-1}B$, can be invertible, where B is the gradient of the constraint equations and M is the mass matrix. It is noted that the solution matrix becomes singular (or ill-conditioned) if two or more constraints become numerically dependent (or almost dependent) upon one another. When that happens, the potential gain in accuracy realized by Baumgarte's stabilization is lost.

Third, Baumgarte's technique requires the solution of an augmented matrix equation that involves the constraint gradient matrix B. This means that whenever additional constraints are introduced or when some of the constraints are relaxed, the matrix profiles of the total-system equations will have to be varied. The task of dynamically varying matrix profiles of the total-system equations can significantly complicate software implementation.

The objective of the present paper is to report a new stabilization technique that is aimed at mitigating the three algorithmic and software difficulties of Baumgarte's technique. First, the new technique induces the errors in the constraint equations to decay according to their principal characteristic

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response time constants; the principal errors in the constraint equations diminish according to their corresponding physical response characteristics. Second, the new technique overcomes the nonconvergence difficulty when two or more constraints become *numerically* dependent. Third, the new technique yields a matrix differential equation for the constraint forces. Hence, the solution of the constraint forces can be carried out in a separate module from that for the primary solution variables (the position vector for the dynamical equations). To this end, the paper is organized as follows.

Section II presents a review of the Lagrangian λ-method[26] for formulating the equations of motion with constraints, including both configuration (*holonomic*) constraints and motion (*nonholonomic*) constraints. An examination of Baumgarte's stabilization for constraints is offered in Sec. III, delineating in detail the three noted algorithmic and software implementation difficulties of the Baumgarte stabilization technique.

Section IV presents a new stabilization technique based on a control synthesis approach. First, we introduce the well-known penalty technique so that the constraint forces are made proportional to violations of the constraint conditions. Second, by tailoring the governing equations of motion and by augmenting the constraint equations with the tailored form of the equations of motion, a stabilized differential form of constraint equation is derived. The resulting stabilized constraint equations are shown to be matrix differential equations with the constraint forces as the primary solution vector, yet possessing no artificial damping as is the case with Baumgarte's technique. Hence, one is left with a set of coupled differential equations of motion in which the generalized displacements and the constraint forces form a conjugate pair of unknowns. It should be mentioned that a similar approach has been successfully utilized for the solution of fluid-structure interaction equations[27] and of fluid-porous soil interaction equations[28] when the interaction equations are partitioned[29,30] and solved in a staggered manner. For this reason, the present method will be called *a staggered stabilization technique*.

Section V reports numerical experiments that illustrate the improved performance of the present staggered stabilization technique.[31] The paper ends with concluding remarks regarding computer implementation issues in production-level MBD simulation modules.

II. Equations of Motion with Constraints

The Lagrangian equations of motion for mechanical systems with constraints can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i + \sum_{k=1}^{m} \lambda_k B_{ki}, \qquad i = 1 \ldots n \tag{1}$$

$$\Phi_k(q, \dot{q}, \ddot{q}) = 0 \tag{2}$$

where L is the system Lagrangian, Φ_k are the constraint conditions imposed either on the subsystem boundaries or on the kinematical relations among the generalized coordinates, q_i are the generalized coordinate components, t is time, $(\dot{\ })$ denotes time differentiation, λ is the Lagrangian multiplier, Q_i is the generalized applied force, and B_{ki} is the ith gradient component of the kth constraint equation, Eq. (2).

In order to focus our subsequent discussions, we specialize Eq. (2) to the holonomic (configuration) case:

$$\Phi_k(q) = 0, \qquad B^h_{ki} = \frac{\partial \Phi_k}{\partial q_i}, \qquad k = 1 \ldots m \tag{3}$$

and to nonholonomic (motion) case:

$$\Phi_k(q, \dot{q}) = 0, \qquad B^{nh}_{ki} = \frac{\partial \Phi_k}{\partial \dot{q}_i}, \qquad k = 1 \ldots m \tag{4}$$

It should be noted that the constraint forces Q^c_i are obtained by

$$Q^c_i = \sum_{k=1}^{m} \lambda_k B_{ki}, \qquad i = 1 \ldots n \tag{5}$$

and *not* by λ_k alone.

Because the two constraints give rise to two different sets of equations of motion, we will treat their time discretization separately. It should be mentioned that a typical MBD system involves both cases; hence, the solution procedure should account for the two constraints concurrently.

Systems with Nonholonomic Constraints Only

When the system involves only nonholonomic constraints, the equations of motion become

$$\begin{bmatrix} M & B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \overline{Q} \\ c \end{Bmatrix} \tag{6}$$

where M is the mass matrix, $\overline{Q}$ consists of the applied force Q, the centrifugal and Coriolis force, and the internal spring force, and c is given by

$$c = -\frac{\partial \Phi}{\partial t} \tag{7}$$

Systems with Holonomic Constraints Only

When the system involves only holonomic constraints, the equations of motion become

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \dot{\lambda} \end{Bmatrix} + \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \overline{Q} \\ c \end{Bmatrix} \tag{8}$$

III. Baumgarte's Stabilization Technique

In Baumgarte's technique, one replaces the second row of Eq. (6) for the case of nonholonomic constraints by

$$\dot{\Phi} + \gamma \Phi = 0 \tag{9}$$

Hence, the right-hand side of the second row of Eq. (6) is modified as

$$c = -\frac{\partial \Phi}{\partial t} - \gamma \Phi \tag{10}$$

Baumgarte sketched a solution scheme that uses the given parabolic stabilization technique as follows. First, the parabolically stabilized equation may be expanded as

$$B\ddot{q} + \frac{\partial \Phi}{\partial t} + \gamma \Phi = 0 \tag{11}$$

By substituting $\ddot{q}$ from the first row of Eq. (6), one obtains for λ in the form

$$(BM^{-1}B^T)\lambda = BM^{-1}\overline{Q} + \frac{\partial \Phi}{\partial t} + \gamma \Phi \tag{12}$$

Hence, λ in the preceding expression can be substituted into the governing equations of motion to yield

$$M\ddot{q} = \overline{Q} - B^T(BM^{-1}B^T)^{-1}\left\{BM^{-1}\overline{Q} + \frac{\partial \Phi}{\partial t} + \gamma \Phi\right\} \tag{13}$$

which can be integrated by an explicit integration formula.

For holonomic cases, he has recommended the following integro-differential form:[16]

$$\dot{\Phi} + 2\gamma\Phi + \gamma^2 \int_{t_0}^{t} \Phi \, dt = 0 \tag{14}$$

so that one obtains

$$c = -\frac{\partial \Phi}{\partial t} - 2\gamma\Phi - \gamma^2 \int_{t_0}^{t} \Phi \, dt \tag{15}$$

In the paper where Baumgarte presented this procedure, no solution scheme was suggested, except that he advocated the adoption of generalized momenta as the primary variables. In the present context of the generalized coordinates q, a plausible implementation of the stabilized integro-differential constraint equations may be realized as follows. First, one integrates the governing equation of motion, Eq. (8a), by an implicit integration formula

$$q^{n+1} = \delta \dot{q}^{n+1} + h_q^n \tag{16}$$

where δ is a formula-dependent stepsize and h_q^n is a historical vector. For example, for the trapezoidal rule, we have

$$\delta = (t_{n+1} - t_n)/2, \qquad h_q^n = q^n + \delta \dot{q}^n \tag{17}$$

Integrating the equations of motion with holonomic constraints once, by the preceding implicit formula, one obtains

$$\dot{q}^{n+1} = \delta M^{-1}(\bar{Q} - B^T \lambda^{n+1}) + h_q^n \tag{18}$$

We now substitute the preceding equation into the stabilized integro-differential constraint equation, Eq. (14), to yield

$$\delta B M^{-1} B^T \lambda^{n+1} = \delta B M^{-1} \bar{Q} + h_q^n + 2\gamma\Phi + \gamma^2 \int_{t_0}^{t} \Phi \, dt \tag{19}$$

After substituting the given expression for λ, one can integrate the resulting equation to obtain q^{n+1} by either an implicit or explicit integration formula. We now offer the following remarks.

Remark 1

Each of the constraints for both the holonomic and nonholonomic cases, $\{\Phi_k,\ k = 1 \dots m\}$, possesses the same parabolic time constant γ, since its solution can be expressed as

$$\Phi_k = C_k e^{-\gamma t}, \qquad k = 1 \dots m \tag{20}$$

Note that the errors committed in each of the constraints also decay with the same single time constant. However, regardless of their physical time constants, the errors in the constraint conditions by the stabilized constraint equations, Eqs. (9) and (14), are forced to decrease at the same rate. Hence, the technique does not take advantage of physically different time constants in order to minimize the errors being accumulated in the constraint equations.

Remark 2

Note that the generalized constraint forces λ in Eq. (12) exist only when the matrix $BM^{-1}B^T$ is not ill-conditioned. Even though the constraints are theoretically independent, such ill-conditioning can occur when two or more constraints become *numerically* nearly dependent, as B is in general state-dependent. If such situations develop, the accuracy of generalized constraint force λ can be considerably degraded, thus leading to a dramatic loss of solution accuracy for q.

Remark 3

From computer implementation considerations, the solution of MBD systems by Baumgarte's technique must be carried out in a tightly coupled program module. Therefore, any change in the number of constraints impacts the matrix structure of the solution procedures, requiring dynamically varying matrix profiles. This can considerably complicate the task of software implementation.

We will now present a new stabilization technique that mitigates the three algorithmic and software implementation difficulties in Baumgarte's stabilization technique pointed out in the preceding remarks.

IV. New Technique: Staggered Stabilization Procedure

In Baumgarte's stabilization technique, as discussed in the preceding section, the objective was to minimize the errors initiated in the constraint condition

$$\Phi = 0 \tag{21}$$

First, the difficulty associated with numerically dependent constraints alluded to in Remark 2 can be overcome by adopting the penalty procedure

$$\lambda = \frac{1}{\epsilon}\Phi, \qquad \epsilon \to 0 \tag{22}$$

as the basic constraint equations instead of Eqs. (3) and (4). It is noted that the penalty procedure as given by Eq. (22) tacitly assumes violations of the constraint condition in actual computations. If one substitutes Eq. (22) into the governing equations of motion, the result becomes

$$M\ddot{q} + \frac{1}{\epsilon} B^T \Phi = \bar{Q} \tag{23}$$

It can be shown that this penalty procedure mitigates nonconvergence difficulties in the constraint conditions. However, its major drawback is that once an error is committed in computing λ, there is no compensation scheme by which the drifting of the numerical solution can be corrected. It is this observation that has led to the development of a staggered stabilization procedure as described in the following paragraphs.

To illustrate the new procedure we will consider the case of nonholonomic constraints. Instead of substituting the penalty expression directly into the governing equations of motion, first we differentiate Eq. (22) once to obtain

$$\dot{\lambda} = \frac{1}{\epsilon}\left(B\dot{q} + \frac{\partial \Phi}{\partial t}\right) \tag{24}$$

where we assume the penalty parameter ϵ to be constant.

Second, we obtain $\ddot{q}$ from Eq. (6a) in the form

$$\ddot{q} = M^{-1}(\bar{Q} - B^T \lambda) \tag{25}$$

and substitute it into Eq. (24) to yield

$$\epsilon\dot{\lambda} + BM^{-1}B^T\lambda = BM^{-1}\bar{Q} + \frac{\partial \Phi}{\partial t} \tag{26}$$

Notice that the homogeneous part of this stabilized equation in terms of the generalized constraint forces λ has the following companion eigenvalue problem:

$$(\gamma + BM^{-1}B^T/\epsilon)\, y = 0 \tag{27}$$

where $\{\gamma_k,\ k = 1 \dots m\}$ are the eigenvalues of the homogeneous operator for the new stabilized constraint equations, Eqs. (26). Since γ_k also dictates how the errors in the constraint equations will diminish with time, the errors committed in the constraint conditions will decay with their corresponding different response time constants. This physically oriented stabilization property of the present technique is in contrast to that of Baumgarte's technique wherein all the error components diminish according to a single time constant.

Third, the new technique enables us to solve for λ from the stabilized differential equation, Eq. (26). Specifically, we now have a set of coupled equations, one for the generalized coordinates q and the other for the generalized constraint forces λ, which are recalled here from Eqs. (6a) and (26) for the case of nonholonomic constraints:

$$\begin{bmatrix} M & 0 \\ 0 & \epsilon \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \dot{\lambda} \end{Bmatrix} + \begin{bmatrix} 0 & B^T \\ 0 & BM^{-1}B^T \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \overline{Q} \\ BM^{-1}\overline{Q} + \dfrac{\partial \Phi}{\partial t} \end{Bmatrix} \tag{28}$$

Note that these coupled equations directly provide the desired differential equations for a conjugate pair of $\lfloor q \ \lambda \rfloor$.

Remark 4

For holonomic constraints, one has several stabilization possibilities. The one we have chosen is to integrate the governing equations of motion once to obtain

$$\dot{q}^n = \delta M^{-1}\left(\overline{Q}^n - B^T \lambda^n\right) + h^n_{\dot{q}} \tag{29}$$

which is substituted into

$$\dot{\lambda} = \frac{1}{\epsilon}\left(B\dot{q} + \frac{\partial \Phi}{\partial t}\right) \tag{30}$$

to yield

$$\epsilon \dot{\lambda}^n + \delta B M^{-1} B^T \lambda^n = B\left(\delta M^{-1}\overline{Q}^n + h^n_{\dot{q}}\right) + \frac{\partial \Phi}{\partial t} \tag{31}$$

Remark 5

It is observed that even if $BM^{-1}B^T$ is almost singular, the new stabilization technique as derived in Eqs. (26) and (31) would not cause numerical difficulty in computing λ since the solution iteration matrix becomes $(\epsilon + \delta BM^{-1}B^T)$ for nonholonomic cases and $(\epsilon + \delta^2 BM^{-1}B^T)$ for holonomic cases.

Remark 6

The present staggered stabilization technique and Baumgarte's technique can be presented in control-synthesis block diagrams, as shown in Figs. 1a and 1b. For nonholonomic constraints, the present technique can be viewed as a combination of gain plus rate feedback stabilization, whereas Baumgarte's technique is seen as a simple gain feedback stabilization. For holonomic constraints, a similar distinction can be observed. The resulting feature of a rate feedback manifested in the present staggered stabilization technique constitutes an important attribute as it copes with the dynamical nature of the problem.

V. Numerical Evaluation

The first problem is a one-bar rigid pendulum problem studied in Ref. 15. The equations of motion consist of both horizontal and vertical trajectories of the pendulum's tip plus one constraint equation for the circular motion of the tip; thus, there are two position variables and one holonomic constraint condition. First, we fix the integration stepsize and carry out the numerical solution by the trapezoidal rule without iteration for both stabilization techniques. Figure 2 shows the errors in the constraint condition for the two techniques. The results show that the present technique yields accuracy about two orders of magnitude higher than that yielded by Baumgarte's technique. In order to gain further insight, the accuracy level in the constraint condition is fixed to be the same (10^{-6}) at each time step and the solution matrix is iterated to satisfy the accuracy requirement. Figure 3 illustrates the number of iterations needed at each step vs time. Note that the average iteration number for the present technique is about four, whereas with Baumgarte's technique it is about six.

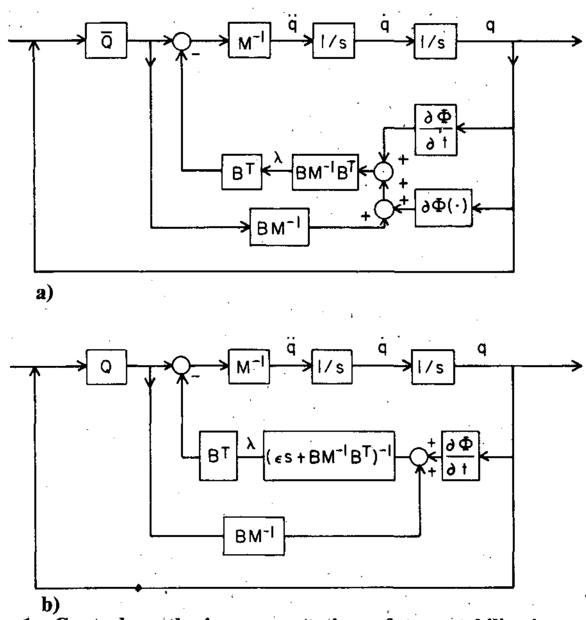

Fig. 1 Control synthesis representation of two stabilization techniques: a) Baumgarte's technique, and b) stabilized technique.

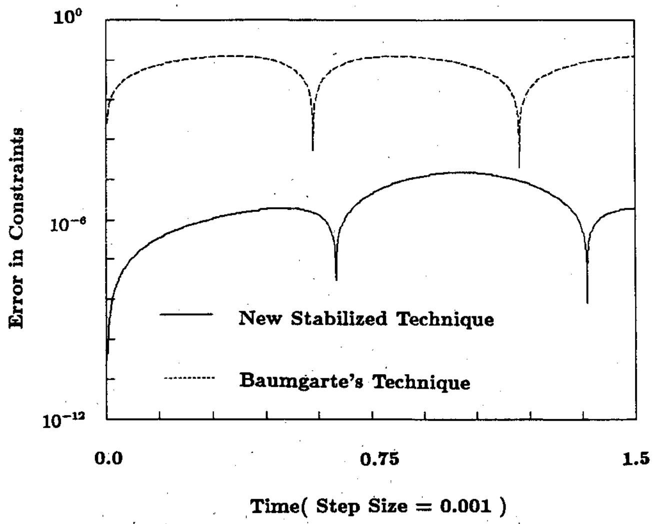

Fig. 2 Errors in constraint with no iteration, performance of two stabilized techniques (single pendulum problem).

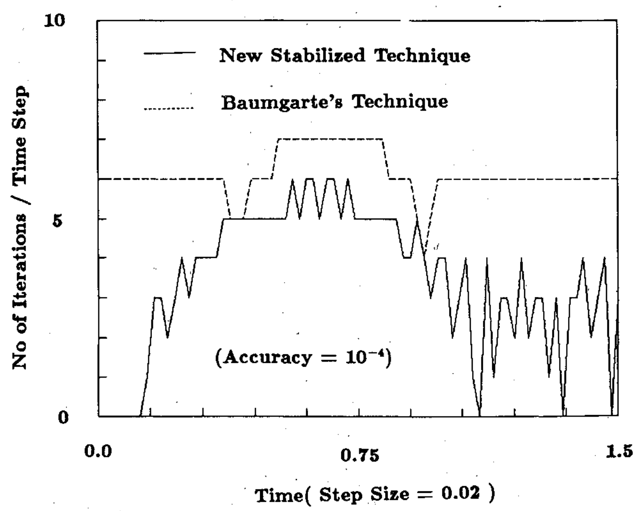

Fig. 3 Number of iterations required for given error tolerance, performance of two stabilized techniques (single pendulum problem).

The second example is a classical crank mechanism whose governing equations of motion are characterized by the following matrices and constraints [see Eqs. (3–8) for their definitions]:

$$M = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & m & \\ & & & m \end{bmatrix} \tag{32}$$

$$\Phi = \begin{Bmatrix} r\cos\theta - (x - l_1\cos\phi) \\ r\sin\theta - (y - l_1\sin\phi) \\ (l - l_1)\sin\phi + y \end{Bmatrix} = 0 \tag{33}$$

$$B^T = \begin{bmatrix} -r\sin\theta & r\cos\theta & 0 \\ -l_1\sin\theta & l_1\cos\phi & (l - l_1)\cos\phi \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{34}$$

and

$$q = \lfloor \theta \quad \phi \quad x \quad y \rfloor^T, \qquad \lambda = \lfloor \lambda_1 \quad \lambda_2 \quad \lambda_3 \rfloor^T$$

$$Q = \{0 \quad 0 \quad 0 \quad -mg\}^T \tag{35}$$

Figure 4 shows the problem definition along with the numerical performance of the two procedures, Baumgarte's technique and the staggered stabilization technique. The performance of the Baumgarte technique and that of the staggered stabilization technique for this problem are also presented in Fig. 4. In carrying out the computations, the trapezoidal rule has been used to time-discretize the equations of motion [Eqs. (2)], the constraints [Eqs. (3)], and their stabilized forms [Eqs. (19) and (28)]. A sufficiently small step increment was used, corresponding to 82 increments for one cycle of the mechanism, with the time increment $h = 0.01$ for the period $T = 0.82$. In order to measure the performance of the two techniques directly, in terms of violation of the constraint conditions vs time during one complete cycle, no iteration was performed at each integration step. In each technique, the three constraint conditions exhibited the same order of accuracy level. Hence, we illustrate only one constraint violation history, i.e., the pin joint constraint between the crank and the connecting rod. Note that the error in the constraint condition for Baumgarte's technique remains about two digits above that with the staggered stabilization technique. In addition, we have experimented with several values of α and β that are required in Baumgarte's technique, and the best parameter choice was found to be $\alpha = \beta = 70$. For the staggered stabilization technique, the penalty parameter chosen was $\epsilon = 10^{-6}$, which yielded an accuracy level about 10^{-5} for the technique.

The third problem tested is a simplified version of the seven-link manipulator deployment problem.[13] The three links are initially folded and, for modeling simplicity, between the two joints is a coil spring that resists a constant deploying force at the tip of the third link. Also, the left-hand end of the first link is fixed through the same coil spring to the wall. These three coil springs are to be *locked up* once the links are deployed straight. The deployment sequence of the manipulator is illustrated in Fig. 5. The time-discretized difference equations both for Baumgarte's technique and the staggered stabilization technique have been solved at each time increment by a Newton-type iterative procedure to meet a specified accuracy level. Hence, the performance of the two techniques can be assessed by the average number of iterations taken per time increment. This is presented in Fig. 6 for the accuracy of 10^{-4}. Notice that the staggered stabilization technique requires on the average about 4.5 iterations per step, whereas Baumgarte's technique requires about 22 iterations per step.

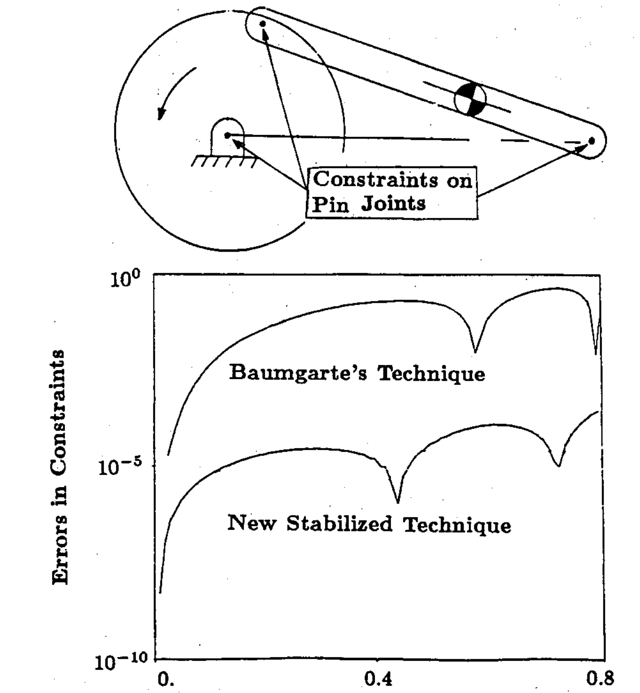

Fig. 4 Errors in pin-joint constraint with no iteration, performance of two techniques.

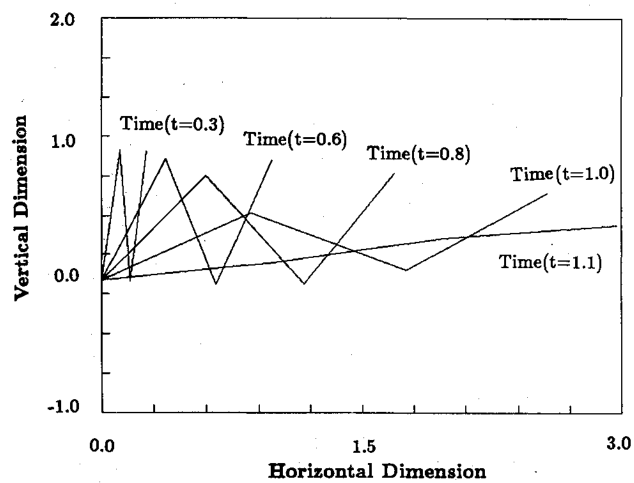

Fig. 5 Deployment of three-link remote manipulator.

Note that Baumgarte's technique fails to converge for time, $t \approx 1.1$, as manifested in Fig. 6 because the rows in B become numerically dependent upon one another when the links are in a straight configuration. This corroborates the theoretical prediction of nonconvergence whenever the solution matrix $BM^{-1}B^T$ for Baumgarte's technique [see Eq. (12)] becomes singular. On the other hand, the staggered stabilization technique still converges within 30 iterations because it overcomes this singularity difficulty, since λ still exists, as can be seen from Eqs. (26) and (31). Although not reported here, the same relative performance has been observed for different accuracy levels, i.e., for the accuracy of 10^{-5} and 10^{-6}.

From the sample test problems, we conclude that the staggered stabilization technique yields both improved accuracy over and greater computational robustness than the Baumgarte technique. In addition, the staggered stabilization technique offers software modularity in that the solution of the constraint force λ can be carried out separately from that of the generalized displacement q. The only data each solution module needs to exchange with the other is a set of vectors, plus a common module to generate the gradient matrix of the

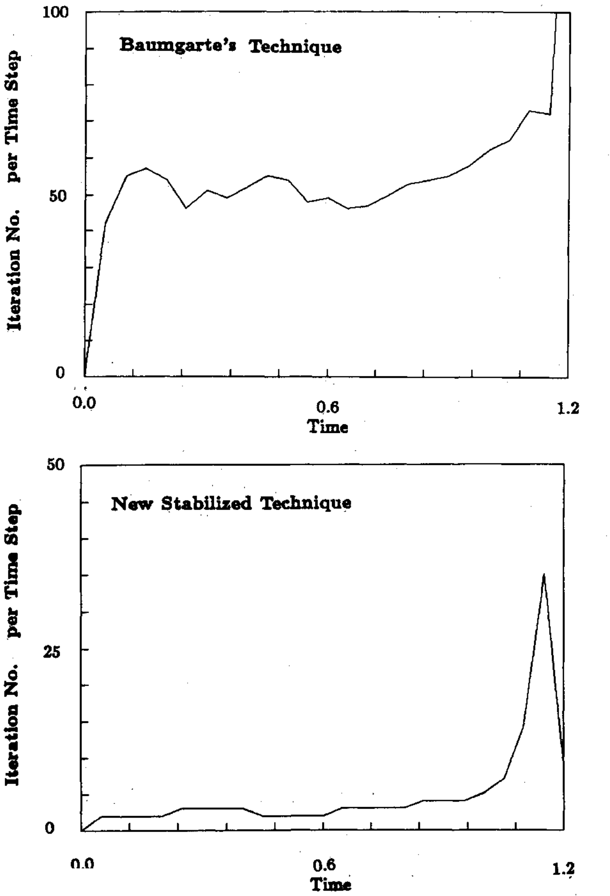

Fig. 6 Performance of two stabilization techniques for three-link manipulator (solution accuracy = 10^{-6}).

constraints, B. However, one should be cautioned not to extrapolate blindly to complex problems the results of the present simple examples. Further judicious experiments are needed in applying the present staggered stabilization technique to complex production-level problems before it can be adopted for general applications in multibody dynamic simulations.

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References

[1]Hooker, W. and Margulies, G., "The Dynamical Attitude Equations for an N-body Satellite," *Journal of Astronautical Science*, Vol. 12, 1965, pp. 123–128.

[2]Roberson, R. and Wittenburg, J., "A Dynamical Formalism for an Arbitrary Number of Interconnected Rigid Bodies with Reference to the Problem of Satellite Attitude Control," *Proceedings of the Third International Congress of Automatic Control*, Butterworth, London, 1965.

[3]Likins, P., "Analytical Dynamics and Nonrigid Spacecraft Simulation," TR 32-1593, Jet Propulsion Laboratory, Pasadena, CA, 1974.

[4]Ho, J., "The Direct Path Method for Deriving the Dynamic Equations of Motion of a Multibody Flexible Spacecraft with Topological Tree Configuration," AIAA Paper 74-786, 1974.

[5]Roberson, R., "A Form of the Translational Dynamical Equations for Relative Motion in Systems of Many Non-Rigid Bodies," *Acta Mechanica*, Vol. 14, 1972, pp. 297–308.

[6]Boland, P., Samin, J., and Willems, P., "Stability Analysis of Interconnected Deformable Bodies in a Topological Tree," *AIAA Journal*, Vol. 12, Aug. 1974, pp. 1025–1030.

[7]De Veubeke, B.F., "The Dynamics of Flexible Bodies," *International Journal of Engineering Science*, Vol. 14, 1976, pp. 895–913.

[8]Keat, J.E., *Dynamical Equations of Body Systems with Application to Space Structure Deployment*, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA, 1983.

[9]Kane, T. and Levinson, D., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, March–April 1980, pp. 99–112.

[10]Bodley, C.S., Devers, A.D., Park, A.C., and Frish, H.P., "A Digital Computer Program for the Dynamic Interaction Simulation of Control and Structures (DISCOS)," NASA TP-1219, May 1978.

[11]*The ADAMS User's Guide*, Mechanical Dynamics, Inc., Ann Arbor, MI, 1979.

[12]Keat, J.E., *Dynamical Equations of Rigid Body Systems with Application to Space Structure Deployment*, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA, 1983.

[13]Housner, J.M., "Convected Transient Analysis for Large Space Structure Maneuver and Deployment," *Proceedings of the 25th Structures, Structural Dynamics and Materials Conference*, Part 2, AIAA New York, 1984, pp. 616–619.

[14]Walton, W.C. and Steeves, E.C., "A New Matrix Theorem and Its Application for Establishing Independent Coordinates for Complex Dynamical Systems with Constraints," NASA TR-R326, 1969.

[15]Baumgarte, J.W., "Stabilization of Constraints and Integrals of Motion in Dynamical Systems," *Computational Methods in Applied Mechanics and Engineering*, Vol. 1, 1972, pp. 1–16.

[16]Baumgarte, J.W., "A New Method of Stabilization for Holonomic Constraints," *Journal of Applied Mechanics*, Vol. 50, 1983, pp. 869–870.

[17]Orlandea, N., Chase, M.A., and Calahan, D.A., "A Sparsity-Oriented Approach to the Dynamic Analysis and Design of Mechanical Systems—Parts I and II," *Transactions of the ASME, Journal of Engineering for Industry, Series B*, Vol. 99, 1977, pp. 773–784.

[18]Lötstedt, P., "On a Penalty Function Method for the Simulation of Mechanical Systems Subject to Constraints," TRITA-NA-7919, Royal Institute of Technology, Stockholm, Sweden, 1979.

[19]Wehage, R.A. and Haug, E.J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems," *ASME Journal of Mechanical Design*, Vol. 104, 1982, pp. 247–255.

[20]Gear, C.W., "Simultaneous Numerical Solution of Differential/Algebraic Equations," *IEEE Transactions on Circuit Theory*, CT-18, 1971, pp. 89–95.

[21]Petzold, L., "Differential/Algebraic Equations are not ODEs," *SIAM Journal of Scientific Statistical Computation*, Vol. 3, 1982, pp. 367–384.

[22]*Penalty-Finite Element Methods in Mechanics,* edited by J.N. Reddy, American Society of Mechanical Engineers, AMD Vol. 51, New York, 1982.

[23]Huston R.L. and Kamman, J.W., "A Discussion on Constraint Equations in Multibody Dynamics," *Mechanical Research Communication*, Vol. 9, 1982, pp. 251–256.

[24]Fuehrer, C. and Wallrapp, O., "A Computer-Oriented Method for Reducing Linearized Multibody System Equations by Incorporating Constraints," *Computational Methods in Applied Mechanics and Engineering*, Vol. 46, 1984, pp.169–175.

[25]Schwertassek, R. and Roberson, R.E., "A State-Space Dynamical Representation for Multibody Mechanical Systems, Part II," *Acta Mechanica*, Vol. 51, 1984, pp. 15–29.

[26]Nikravesh, P.E., "Some Methods for Dynamic Analysis of Constrained Mechanical Systems: A Survey," *Computer Aided Analysis and Optimization of Mechanical System Dynamics*, edited by E.J. Haug, NATO ASI Series, F9, Springer-Verlag, Berlin, 1984, pp. 351–367.

[27]Lanczos, L., *The Variational Principles of Mechanics,* 4th ed., University of Toronto Press, Toronto, 1970, pp. 141–147.

[28]Park, K.C., Felippa, C.A., and DeRuntz, J.A., "Stabilization of Staggered Solution Procedures for Fluid-Structure Interaction Analysis," *Computational Methods for Fluid-Structure Interaction Problems*, edited by T. Belytschko and T.L. Geers, ASME, AMD Vol. 26, New York, 1977, pp. 95–124.

[29]Park, K.C., "Partitioned Transient Analysis Procedures for Coupled-Field Problems: Stability Analysis," *Journal of Applied Mechanics*, Vol. 47, 1980, pp. 370–376.

[30]Park, K.C., "Stabilization of Partitioned Solution Procedures for Pore Fluid-Soil Interaction Analysis," *International Journal of Numerical Methods in Engineering*, Vol. 19, 1983, pp. 1669–1673.

[31]Park, K.C., "Stabilization of Computational Procedures for Constrained Dynamical Systems: Formulation," AIAA Paper 86-0926, May 1986.